

# ON THE TORSION IN THE COHOMOLOGY OF ARITHMETIC HYPERBOLIC 3-MANIFOLDS

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**ABSTRACT.** In this paper we consider the cohomology of a closed arithmetic hyperbolic 3-manifold with coefficients in the local system defined by the even symmetric powers of the standard representation of  $\mathrm{SL}(2, \mathbb{C})$ . The cohomology is defined over the integers and is a finite abelian group. We show that the order of the 2nd cohomology grows exponentially as the local system grows. We also consider the twisted Ruelle zeta function of a closed arithmetic hyperbolic 3-manifold and we express the leading coefficient of its Laurent expansion at the origin in terms of the orders of the torsion subgroups of the cohomology.

## 1. INTRODUCTION

Let  $\mathbf{G}$  be a semi-simple connected algebraic group over  $\mathbb{Q}$ ,  $K$  a maximal compact subgroup of its group of real points  $G = \mathbf{G}(\mathbb{R})$  and  $S = G/K$  the associated Riemannian symmetric space. Let  $\Gamma \subset \mathbf{G}(\mathbb{Q})$  be an arithmetic subgroup and  $X = \Gamma \backslash S$  the corresponding locally symmetric space. Let  $\rho: \mathbf{G} \rightarrow \mathrm{GL}(V)$  be a rational representation of  $\mathbf{G}$  on a  $\mathbb{Q}$ -vector space  $V$ . Then there exists a lattice  $M \subset V$  which is stable under  $\Gamma$ . Let  $\mathcal{M}$  be the associated local system of free  $\mathbb{Z}$ -modules over  $X$  defined by the  $\Gamma$ -module  $M$  and let  $H^*(X, \mathcal{M})$  be the corresponding sheaf cohomology. Since  $X$  has the homotopy type of a finite  $CW$ -complex,  $H^*(X, \mathcal{M})$  are finitely generated abelian groups. The cohomology of arithmetic groups has important connections to the theory of automorphic forms and number theory [Sch]. In this respect,  $H^*(X, \mathcal{M} \otimes \mathbb{C})$  has been studied to a great extent. Much less is known about the torsion subgroup  $H^*(X, \mathcal{M})_{\mathrm{tors}}$ . In a recent paper, [BV] Bergeron and Venkatesh studied the growth of  $|H^j(\Gamma_N \backslash S, \mathcal{M})_{\mathrm{tors}}|$  as  $N \rightarrow \infty$ , where  $\{\Gamma_N\}$  is a decreasing sequence of normal subgroups of finite index of  $\Gamma$  with trivial intersection. This is motivated by conjectures that torsion classes in the cohomology of arithmetic groups should have arithmetic significance [AS], [ADP]. In this paper we consider a similar problem in a different aspect. We fix the discrete group and vary the local system. We also restrict attention to the case of hyperbolic 3-manifolds. However, we expect that the results hold in greater generality.

Now we explain our results in more detail. Let  $\mathbb{H}^3 = \mathrm{SL}(2, \mathbb{C})/\mathrm{SU}(2)$  be the 3-dimensional hyperbolic space and let  $\Gamma \subset \mathrm{SL}(2, \mathbb{C})$  be a discrete torsion free co-compact subgroup. Then  $X = \Gamma \backslash \mathbb{H}^3$  is a compact oriented hyperbolic 3-manifold. We are interested in arithmetic subgroups  $\Gamma$  which are derived from a quaternion division algebra  $D$  over an imaginary

quadratic number field  $F$ . The division algebra  $D$  determines an algebraic group  $\mathrm{SL}_1(D)$  over  $F$  which is an inner form of  $\mathrm{SL}_2/F$ . Moreover, the group of its  $F$ -rational points  $\mathrm{SL}_1(D)(F)$  is equal to

$$D^1 := \{x \in D : N(x) = 1\},$$

where  $N(x) = x\bar{x}$  denotes the norm of  $x \in D$ . Let  $\mathfrak{o} \subset D$  be an order in  $D$  and let  $\mathfrak{o}^1 = \mathfrak{o} \cap D^1$ . Then  $\mathfrak{o}^1$  is a discrete subgroup of  $D^1 = \mathrm{SL}_1(D)(F)$ . The quaternion division algebra  $D$  splits over  $\mathbb{C}$ , i.e., there is an isomorphism  $\varphi : D \otimes_F \mathbb{C} \rightarrow M_2(\mathbb{C})$  of  $\mathbb{C}$ -algebras. Let  $\Gamma := \varphi(\mathfrak{o}^1)$ . Then  $\Gamma$  is a cocompact arithmetic subgroup of  $\mathrm{SL}(2, \mathbb{C})$ . For  $n \in \mathbb{N}$  let  $V(n) = S^n(F^2)$  be the  $n$ -th symmetric power of  $F^2$  and let  $\mathrm{Sym}^n$  be the  $n$ -th symmetric power representation of  $\mathrm{SL}_2/F$  on  $V(n)$ . It follows from Galois descent that for every even  $n$  there is a rational representation

$$\mu_n : \mathrm{SL}_1(D)/F \rightarrow \mathrm{GL}(V(n))$$

which is equivalent to  $\mathrm{Sym}^n$  over  $\overline{F}$ . Using this representation it follows that for each even  $n$  there is a lattice  $M_n \subset V(n)$  which is stable under  $\Gamma$  with respect to  $\mathrm{Sym}^n$ . Let  $\mathcal{M}_n$  be the associated local system of free  $\mathbb{Z}$ -modules over  $X$ . Then  $H^*(X, \mathcal{M}_n \otimes \mathbb{R}) = 0$ . Thus  $H^*(X, \mathcal{M}_n)$  is a torsion group. Let  $|H^p(X, \mathcal{M}_n)|$  denote the order of  $H^p(X, \mathcal{M}_n)$ . The purpose of this paper is to study the behavior of  $\log |H^p(X, \mathcal{M}_n)|$  as  $n \rightarrow \infty$ . Our main result is the following theorem.

**Theorem 1.1.** *For every choice of  $\Gamma$ -stable lattices  $M_{2k}$  in  $S^{2k}(\mathbb{C}^2)$  we have*

$$(1.1) \quad \lim_{k \rightarrow \infty} \frac{\log |H^2(X, \mathcal{M}_{2k})|}{k^2} = \frac{2}{\pi} \mathrm{vol}(X).$$

Furthermore, for  $p = 1, 3$  we have

$$(1.2) \quad \log |H^p(X, \mathcal{M}_{2k})| \ll k \log k$$

uniformly over all choices of lattices  $M_{2k}$ .

Note that the left hand side of (1.1) is a pure combinatorial invariant of  $X$ . So at first sight it looks surprising that the volume appears on the right hand side. However, this is no contraction, since we know by the Mostow-Prasad rigidity theorem that the volume of a hyperbolic manifold is a topological invariant.

The proof of (1.1) is a consequence of the following theorem combined with (1.2).

**Theorem 1.2.** *The alternating sum of  $\log |H^p(X, \mathcal{M}_{2k})|$  is independent of the choice of a  $\Gamma$ -stable lattice  $M_{2k}$  in  $S^{2k}(\mathbb{C}^2)$  and we have*

$$(1.3) \quad \sum_{p=1}^3 (-1)^p \log |H^p(X, \mathcal{M}_{2k})| = \frac{2}{\pi} \mathrm{vol}(X) k^2 + O(k)$$

as  $k \rightarrow \infty$ .

The fact that  $\sum_{p=1}^3 (-1)^p \log |H^p(X, \mathcal{M}_{2k})|$  is independent of the choice of the  $\Gamma$ -stable lattice  $M_{2k}$  follows from the proof of (1.3), but it can also be seen in more elementary way. See the remark following (5.31).

More generally for  $m, n \in \mathbb{N}$  even, we may consider the local system

$$\mathcal{M}_{n,m} = \mathcal{M}_n \otimes \overline{\mathcal{M}}_m.$$

where  $\overline{\mathcal{M}}_n$  is the local system attached to the complex conjugated lattice  $\overline{M}_n$  of  $M_n$ . If  $n \neq m$ , then nothing changes. We still have  $H^*(X, \mathcal{M}_{n,m} \otimes \mathbb{R}) = 0$ . Thus for  $n \neq m$ ,  $H^*(X, \mathcal{M}_{n,m})$  is a torsion group and there is an asymptotic formula similar to (1.1) as  $m \rightarrow \infty$  or  $n \rightarrow \infty$ .

In [BV] Bergeron and Venkatesh established results of similar nature but in a different aspect. They study the growth of the torsion in the cohomology for a fixed local system as the lattice varies in a decreasing sequence of congruence subgroups. Again the volume of the locally symmetric space appears as the main ingredient of the asymptotic formulas.

Our next result is related to the Ruelle zeta function of  $X$ . In [De] Deninger discussed a geometric analogue of Lichtenbaum's conjectures in the context of Ruelle zeta functions attached to certain dynamical systems. We establish a result of similar nature for the Ruelle zeta function of a compact arithmetic hyperbolic 3-manifold.

The Ruelle zeta function is a dynamical zeta function attached to the geodesic flow on the unit tangent bundle of  $X$ . We recall its definition. Let  $\chi: \Gamma \rightarrow \mathrm{GL}(V)$  be a representation on a finite-dimensional complex vector space. Given  $\gamma \in \Gamma$ , denote by  $[\gamma]$  the  $\Gamma$ -conjugacy class of  $\gamma$ . For  $\gamma \in \Gamma \setminus \{e\}$  let  $\ell(\gamma)$  be the length of the unique closed geodesic that corresponds to  $[\gamma]$ . Then the twisted Ruelle zeta function is defined as

$$(1.4) \quad R(s; \chi) := \prod_{\substack{[\gamma] \neq e \\ \text{prime}}} \det(\mathrm{I} - \chi(\gamma) e^{-s\ell(\gamma)})^{-1}.$$

The product runs over all nontrivial primitive conjugacy classes. The infinite product converges in some half-plane  $\mathrm{Re}(s) > c$  and admits a meromorphic extension to  $\mathbb{C}$  [Fr2, Sect. 7]. We note that the definition (1.4) differs from the usual one by the exponent  $-1$ .

We consider the special case where  $\chi$  is the restriction to  $\Gamma$  of a finite-dimensional representation  $\rho$  of  $\mathrm{SL}(2, \mathbb{C})$ . We denote the associated twisted Ruelle zeta function by  $R(s; \rho)$ . Note that the irreducible finite-dimensional representations of  $\mathrm{SL}(2, \mathbb{C})$ , regarded as real Lie group, are given by

$$(1.5) \quad \rho_{m,n} := \mathrm{Sym}^m \otimes \overline{\mathrm{Sym}^n}, \quad m, n \in \mathbb{N},$$

[Kn, p. 32]. Our interest is in the behavior of  $R(s; \rho)$  at  $s = 0$ . To describe it we need to introduce the regulator associated to the free part of the cohomology of a local system of free  $\mathbb{Z}$ -modules.

Let  $\rho: \mathrm{SL}(2, \mathbb{C}) \rightarrow \mathrm{GL}(V)$  be a finite-dimensional real representation of  $\mathrm{SL}(2, \mathbb{C})$  and assume that there exists a lattice  $M \subset V$  which is stable under  $\Gamma$ . Let  $\mathcal{M}$  be the associated

local system. Let  $E$  be the flat vector bundle over  $X$  attached to the restriction of  $\rho$  to  $\Gamma$ . By [MM, Lemma 3.1], the bundle  $E$  can be equipped with a canonical fibre metric. Let  $\mathcal{H}^*(X; E)$  be the space of  $E$ -valued harmonic forms on  $X$  with respect to this metric in  $E$  and the hyperbolic metric on  $X$ . There is a canonical isomorphism

$$\mathcal{H}^*(X; E) \cong H^*(X, \mathcal{M} \otimes \mathbb{R}).$$

We equip  $H^*(X; \mathcal{M} \otimes \mathbb{R})$  with the inner product  $\langle \cdot, \cdot \rangle$  induced by the  $L^2$ -metric on  $\mathcal{H}^*(X; E)$ . Let  $H^*(X; \mathcal{M})_{\text{free}} = H^*(X; \mathcal{M}) / H^*(X; \mathcal{M})_{\text{tors}}$ . We identify  $H^*(X; \mathcal{M})_{\text{free}}$  with a subgroup of  $H^*(X; \mathcal{M} \otimes \mathbb{R})$ . For each  $p = 0, 1, 2, 3$  choose a basis  $a_1, \dots, a_{r_p}$  of  $H^*(X; \mathcal{M})_{\text{free}}$  and let  $G_p(\mathcal{M})$  be the Gram matrix of this basis with respect to the inner product  $\langle \cdot, \cdot \rangle$ . Put  $R_p(\mathcal{M}) := \sqrt{|\det G_p(\mathcal{M})|}$ . Then the regulator is defined as

$$R(\mathcal{M}) := \prod_{p=0}^3 R_p(\mathcal{M})^{(-1)^p}.$$

We can now state our result which describes the behavior of the Ruelle zeta function at the origin.

**Theorem 1.3.** *Let  $X = \Gamma \backslash \mathbb{H}^3$  be a compact oriented arithmetic hyperbolic 3-manifold. Let  $\rho: \text{SL}(2, \mathbb{C}) \rightarrow \text{GL}(V)$  be an irreducible finite-dimensional representation of  $\text{SL}(2, \mathbb{C})$ . Let  $M \subset V$  be a lattice which is stable under  $\Gamma$  and denote by  $\mathcal{M}$  the associated local system of free  $\mathbb{Z}$ -modules over  $X$ . Let  $R(s; \rho)$  be the twisted Ruelle zeta function. Then we have*

1) *If  $\rho \neq 1$ , then the order of  $R(s; \rho)$  at  $s = 0$  is given by*

$$(1.6) \quad \text{ord}_{s=0} R(s; \rho) = \sum_{q=1}^3 (-1)^q q \text{rk } H^q(X; \mathcal{M}).$$

*If  $\rho = 1$ , then the order equals  $2 \dim H^1(X, \mathbb{R}) - 4$ .*

2) *Let  $R^*(0; \rho)$  be the leading coefficient of the Laurent expansion of  $R(s; \rho)$  at  $s = 0$ . We have*

$$(1.7) \quad |R^*(0; \rho)| = R(\mathcal{M})^{-1} \cdot \prod_{q=0}^3 |H^q(X; \mathcal{M})_{\text{tors}}|^{(-1)^q}.$$

*Moreover, if  $\rho = \rho_{m,m}$  for some  $m \in \mathbb{N}_0$ , then (1.7) holds for  $R^*(0; \rho)$ .*

Recall that the irreducible finite-dimensional representations of  $\text{SL}(2, \mathbb{C})$  are given by (1.5). For each even  $n$  there is a  $\Gamma$ -stable lattice in  $S^n(\mathbb{C}^2)$ . Therefore, if  $\rho = \rho_{m,n}$  with  $m$  and  $n$  even, there exist  $\Gamma$ -stable lattices in the space of the representation.

We note that there is a formal analogy of (1.6) and (1.7) to Lichtenbaum's conjectures on special values of Hasse-Weil zeta functions of algebraic varieties [Li1], [Li2]. To make this statement more transparent, we recall some details. Let  $\mathcal{X}$  be a regular scheme which is separated and of finite type over  $\text{Spec}(\mathbb{Z})$ . Let  $\zeta_{\mathcal{X}}(s)$  be the Hasse-Weil zeta function of  $\mathcal{X}$ . It is given by an Euler product that converges in some half-plane  $\text{Re}(s) > c$ . The

Euler product is expected to have a meromorphic extension to the whole complex plane. This is known in some cases. Lichtenbaum's conjectures are concerned with the behavior of  $\zeta_{\mathcal{X}}(s)$  at  $s = 0$ . First of all, Lichtenbaum conjectures the existence of a certain new cohomology theory for schemes over  $\mathbb{Z}$ , called “Weil-étale” cohomology. Let  $H_c^p(\mathcal{X}, \mathbb{Z})$  be the  $p$ -th “Weil-étale” cohomology group of  $\mathcal{X}$  with compact supports and coefficients in  $\mathbb{Z}$ . Then the conjectures of Lichtenbaum are the following statements: 1) The groups  $H_c^p(\mathcal{X}, \mathbb{Z})$  are finitely generated and vanish for  $p > 2 \dim \mathcal{X} + 1$ . 2) The order of  $\zeta_{\mathcal{X}}(s)$  at  $s = 0$  is given by

$$\text{ord}_{s=0} \zeta_{\mathcal{X}}(s) = \sum_p (-1)^p p \text{rk } H_c^p(\mathcal{X}, \mathbb{Z}).$$

3) The leading coefficient  $\zeta_{\mathcal{X}}^*(0)$  of the Laurent expansion of  $\zeta_{\mathcal{X}}(s)$  satisfies

$$\zeta_{\mathcal{X}}^*(0) = R^{-1} \cdot \prod_p |H_c^p(\mathcal{X}, \mathbb{Z})_{\text{tors}}|^{(-1)^p},$$

where  $R$  is the Reidemeister torsion of some acyclic complex associated to  $H_c^*(\mathcal{X}, \mathbb{R})$ , equipped with volume forms determined by the isomorphism  $H_c^*(\mathcal{X}, \mathbb{R}) = H_c^*(\mathcal{X}, \mathbb{Z}) \otimes \mathbb{R}$  and a basis of  $H_c^*(\mathcal{X}, \mathbb{Z})_{\text{free}}$ . We note that Denniger [Ch] first discussed a geometric analogue of the Lichtenbaum conjectures in the context of dynamical systems.

Our approach to prove our main results is based on the study of the Reidemeister torsion of  $X$ . Let  $\rho_{2k}$  be the representation of  $\Gamma$  obtained by the restriction of  $\text{Sym}^{2k}: \text{SL}(2, \mathbb{C}) \rightarrow \text{GL}(S^{2k}(\mathbb{C}^2))$  to  $\Gamma$ . Let  $\rho_{2k}^{\mathbb{R}}$  be the underlying real representation. Denote by  $\tau_X(\rho_{2k}^{\mathbb{R}})$  be the Reidemeister torsion of  $X$  with respect to  $\rho_{2k}^{\mathbb{R}}$  (see section 2 for its definition). Then the Reidemeister torsion satisfies

$$\tau_X(\rho_{2k}^{\mathbb{R}}) = \prod_{p=1}^3 |H^p(X, \mathcal{M}_{2k})|^{(-1)^{p+1}}.$$

This equality was first noted by Cheeger [Ch, (1.4)]. Now we apply [Mu2, Corollary 1.2] which describes the asymptotic behavior of  $\log \tau_X(\rho_{2k}^{\mathbb{R}})$  as  $k \rightarrow \infty$ . This implies Theorem 1.2. To estimate  $\log |H^p(X, \mathcal{M}_{2k})|$  for  $p = 1, 3$ , we use that  $H^3(X, \mathcal{M}_{2k})$  is isomorphic to the space  $(M_{2k})_{\Gamma}$  of coinvariants. To bound  $(M_{2k})_{\Gamma}$  we can work locally. This leads to (1.2). Together with Theorem 1.2 we obtain (1.1), which proves Theorem 1.1.

To prove Theorem 1.3, we use [Mu2, Theorem 1.5]. Using this theorem we obtain the statement about the order of  $R(s; \rho)$  at  $s = 0$ . Moreover, it follows that the leading coefficient of the Laurent expansion of  $R(s; \rho)$  at  $s = 0$  equals  $T_X(\rho; h)^{-2}$ , where  $T_X(\rho; h)$  is the Ray-Singer analytic torsion of  $X$  and  $\rho|_{\Gamma}$  with respect to the canonical fibre metric in the flat bundle  $E$  mentioned above. By [Mu1, Theorem 1], the analytic torsion equals the Reidemeister torsion  $\tau_X(\rho; h)$ . If there exists a  $\Gamma$ -stable lattice in  $V$ , then the Reidemeister torsion satisfies

$$(1.8) \quad \tau_X(\rho; h) = R(\mathcal{M})^{-1} \prod_{p=0}^3 |H^p(X, \mathcal{M})_{\text{tors}}|^{(-1)^{p+1}}.$$

Combining these facts, we obtain Theorem 1.3.

The paper is organized as follows. In section 2 we discuss the relation between Reidemeister torsion and cohomology, if the chain complex is defined over the integers. In particular, we prove (1.8). In section 3 we collect a number of facts about cocompact arithmetic subgroups of  $\mathrm{SL}(2, \mathbb{C})$  which are derived from quaternion division algebras. In particular, we prove that the even symmetric powers contain  $\Gamma$ -stable lattices. In section 4 we prove (1.2). In the final section 5 we prove our main results.

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## 2. REIDEMEISTER TORSION AND COHOMOLOGY

We recall some facts about Reidemeister torsion. Let  $V$  be a finite-dimensional real vector space of dimension  $n$ . Set  $\det V = \Lambda^n(V)$ . A volume element in  $V$  is a nonzero element  $\omega \in \det V$ . Any volume element determines an isomorphism  $\det V \cong \mathbb{R}$ . Furthermore, note that a volume element is given by  $\omega = e_1 \wedge e_2 \wedge \cdots \wedge e_n$  for some basis  $e_1, \dots, e_n$  of  $V$ .

Let

$$(2.9) \quad C^*: 0 \rightarrow C^0 \xrightarrow{d_0} C^1 \xrightarrow{d_1} \cdots \xrightarrow{d_{n-1}} C^n \rightarrow 0$$

be a complex of finite dimensional  $\mathbb{R}$ -vector spaces. Let

$$B^j := d_{j-1}(C^{j-1}), \quad Z^j := \ker(d_j)$$

and denote by  $H^j(C^*) := Z^j/B^j$  the  $j$ -th cohomology group of  $C^*$ . Assume that for each  $j$  we are given volume elements  $\omega_j \in \det C^j$  and  $\mu_j \in \det H^j(C^*)$ . Let  $\omega = (\omega_0, \dots, \omega_n)$  and  $\mu = (\mu_0, \dots, \mu_n)$ . Then the Reidemeister torsion  $\tau(C^*, \omega, \mu) \in \mathbb{R}^+$  of the complex  $C^*$  is defined as a certain ratio of volumes (see [Mi]). In [RS] Ray and Singer gave an equivalent definition in terms of the combinatorial Laplacian which we recall next. In each  $C^j$  we choose an inner product  $\langle \cdot, \cdot \rangle_j$  with volume element  $\omega_j$ . Let

$$d_{j+1}^*: C^{j+1} \rightarrow C^j$$

be the adjoint operator to  $d_j$  with respect to the inner products in  $C^j$  and  $C^{j+1}$ , respectively. Define the combinatorial Laplacian by

$$(2.10) \quad \Delta_j^{(c)} = d_{j+1}^* d_j + d_{j-1} d_j^*.$$

Then  $\Delta_j^{(c)}$  is a symmetric non-negative operator in  $C^j$ . We have the combinatorial Hodge decomposition

$$(2.11) \quad C^j = \ker(\Delta_j^{(c)}) \oplus d_{j-1}(C^{j-1}) \oplus d_{j+1}^*(C^{j+1}).$$

It implies that there is a canonical isomorphism

$$(2.12) \quad \ker(\Delta_j^{(c)}) \xrightarrow{\cong} H^j(C^*).$$

The inner product in  $C^j$  restricts to an inner product in  $\ker(\Delta_j^{(c)})$ . Using the isomorphism (2.12) we get an inner product  $\langle \cdot, \cdot \rangle$  in  $H^j(C^*)$ . Let  $h_1, \dots, h_{d_j} \in H^j(C^*)$  be a basis such that  $\mu_j = h_1 \wedge \dots \wedge h_{d_j}$ . Let  $G_j$  be the Gram matrix with entries  $\langle h_k, h_l \rangle_j$ ,  $1 \leq k, l \leq d_j$ . Put

$$V(\mu_j) = \sqrt{|\det G_j|}.$$

Denote by  $\det' \Delta_j^{(c)}$  the product of the nonzero eigenvalues of  $\Delta_j^{(c)}$ . Then a slight generalization of [RS, Proposition 1.7] gives

**Lemma 2.1.** *We have*

$$\tau(C^*, \omega, \mu) = \prod_{j=0}^n V(\mu_j)^{(-1)^j} \cdot \prod_{j=1}^n \left( \det' \Delta_j^{(c)} \right)^{(-1)^{j+1} j/2}.$$

Now let

$$(2.13) \quad A^*: 0 \longrightarrow A^0 \xrightarrow{d_0} A^1 \xrightarrow{d_1} \dots \xrightarrow{d_{n-1}} A^n \longrightarrow 0$$

be a complex of free finite rank  $\mathbb{Z}$ -modules and set  $C^j := A^j \otimes \mathbb{R}$ . Then we get a complex (2.9) of  $\mathbb{R}$ -vector spaces such that each  $C^j$  has a preferred equivalence class of bases coming from bases of  $A^j$ . For any two such bases the matrix of change from one to the other is an invertible matrix with integral entries. Thus  $C^j$  is endowed with a canonical volume element  $\omega_j$ . Furthermore, with respect to any of the bases coming from  $A^j$ , the coboundary operator  $d_j$  is represented by a matrix with integral entries. Let  $H^j(A^*)$  denote the  $j$ -th cohomology group of the complex (2.13). Let  $H^j(A^*)_{\text{tors}}$  be the torsion subgroup of  $H^j(A^*)$  and let  $H^j(A^*)_{\text{free}} = H^j(A^*)/H^j(A^*)_{\text{tors}}$ . We identify  $H^j(A^*)_{\text{free}}$  with a subgroup of  $H^j(C^*)$ . Let  $\mu_j$  be a volume element of  $H^j(C^*)$ . Put

$$(2.14) \quad R(\mu_j) := \text{vol}(H^j(C^*)/H^j(A^*)_{\text{free}}),$$

where the volume is computed with respect to  $\mu_j$ . Denote by  $\tau(C^*, \mu)$  the Reidemeister torsion of  $C^* = A^* \otimes \mathbb{R}$  with respect to the canonical volume element  $\omega$ , which is determined by  $A^*$ , and the volume element  $\mu \in \prod_{j=0}^n \det H^j(C^*)$ . Finally, denote by  $|H^j(A^*)_{\text{tors}}|$  the order of the finite group  $H^j(A^*)_{\text{tors}}$ . Then the following elementary lemma which describes the relation between Reidemeister torsion and the torsion of the cohomology of the complex  $A^*$  is proved in [BV, section 2].

**Lemma 2.2.** *We have*

$$(2.15) \quad \tau(C^*, \mu) = \prod_{j=0}^n R(\mu_j)^{(-1)^j} \cdot \prod_{j=0}^n |H^j(A^*)_{\text{tors}}|^{(-1)^{j+1}}.$$

If  $\mu_j$  is the volume element associated to the equivalence class of bases of  $H^j(C^*)$  coming from bases of the lattice  $H^j(A^*)_{\text{free}}$ , we have  $R(\mu_j) = 1$ . In this case (2.15) was first stated in [Ch, (1.4)] and proved for the case where  $A^*$  is acyclic.

We now turn to the geometric situation. Let  $X$  be compact  $n$ -dimensional Riemannian manifold. Choose a base point  $x_0 \in X$ . Let  $\Gamma := \pi_1(X, x_0)$  be the fundamental group of  $X$

with respect to  $x_0$ , acting on the universal covering  $\tilde{X}$  of  $X$  as deck transformations. Let  $V$  be a finite-dimensional real vector space and let

$$\chi: \Gamma \rightarrow \mathrm{GL}(V)$$

be a representation of  $\Gamma$  on  $V$ . It defines a flat vector bundle  $E$  over  $X$ .

Fix a smooth triangulation  $K$  of  $X$ . Let  $\tilde{K}$  be the lift of  $K$  to a smooth triangulation of  $\tilde{X}$ . We think of  $K$  as being embedded as a fundamental domain in  $\tilde{K}$ , so that  $\tilde{K}$  is the union of the translates of  $K$  under  $\Gamma$ . Let  $C^q(\tilde{K}; \mathbb{Z})$  be the cochain group generated by the  $q$ -simplexes of  $\tilde{K}$ . Then  $C^q(\tilde{K}; \mathbb{Z})$  is a  $\mathbb{Z}[\Gamma]$ -module. Define the twisted cochain group  $C^q(K; V)$  by

$$C^q(K; V) := C^q(\tilde{K}; \mathbb{Z}) \otimes_{\mathbb{Z}[\Gamma]} V.$$

From the cochain complex  $C^*(\tilde{K}, \mathbb{Z})$  we get the twisted cochain complex

$$(2.16) \quad C^*(K; V): 0 \rightarrow C^0(K; V) \xrightarrow{d_0} C^1(K; V) \xrightarrow{d_1} \dots \xrightarrow{d_{n-1}} C^n(K; V) \rightarrow 0.$$

Let  $H^*(K; V)$  be the cohomology groups. They are independent of  $K$  and will be denoted by  $H^*(X; V)$ . The cohomology can also be computed by a different complex. Let  $C^q(K; E)$  be the set of  $E$ -valued  $q$ -cochains [Wh, Chapt. VI]. An element of  $C^q(K; E)$  is a function that assigns to each  $q$ -simplex a section of  $E$  on that simplex. The corresponding cochain complex of finite-dimensional  $\mathbb{R}$ -vector spaces computes the cohomology groups  $H^*(K; E) = H^*(X; E)$ . By [Wh, p. 278] there is canonical isomorphisms

$$H^*(X; V) \cong H^*(X; E).$$

Assume that a volume element  $\theta \in \det V$  is given. Let  $\sigma_j^q$ ,  $j = 1, \dots, r_q$ , be the oriented  $q$ -simplexes of  $K$  considered as preferred bases of the  $\mathbb{Z}[\Gamma]$ -module  $C^q(\tilde{K}; \mathbb{Z})$ . Let  $e_1, \dots, e_m$  be a basis of  $V$  such that  $\theta = \pm e_1 \wedge \dots \wedge e_m$ . Then  $\{\sigma_j^q \otimes e_k: j = 1, \dots, r_q, k = 1, \dots, m\}$  is a preferred basis of  $C^q(K; V)$ . It defines a volume element  $\omega_q \in \det C^q(K; V)$ . The volume element depends on several choices [Mu2, p. 727]. Now assume that  $\chi$  is unimodular, i.e., we have

$$|\det \chi(\gamma)| = 1, \quad \forall \gamma \in \Gamma.$$

Then  $\omega$  is unique up to sign. Let  $\mu \in \det H^*(X; V)$  be a volume element. Then we can define the Reidemeister torsion  $\tau(C^*(K; V); \omega, \mu)$ . It still depends on  $\theta \in \det V$ . However, if the Euler characteristic of  $X$  vanishes, then it is also independent of  $\theta$  [Mu1, p. 727]. It is well known that  $\tau(C^*(K; V); \omega, \mu)$  is invariant under subdivision [Wh]. Since any two smooth triangulations of  $X$  have a common subdivision,  $\tau(C^*(K; V); \omega, \mu)$  is independent of  $K$ . Therefore we may put

$$\tau_X(\chi; \mu) := \tau(C^*(K; V); \omega, \mu).$$

Now pick a fibre metric  $h$  on  $E$ . Let  $\mathcal{H}^*(X; E)$  be the space of  $E$ -valued harmonic forms on  $X$ . Then we have the Hodge-de Rham isomorphism

$$\mathcal{H}^*(X; E) \cong H^*(X; E) \cong H^*(X; V).$$



Using this isomorphism, we get an inner product in  $H^*(X; V)$  and a corresponding volume element  $\mu_h \in \det H^*(X; V)$ . Let

$$\tau_X(\chi; h) := \tau_X(\chi, \mu_h).$$

Now assume that there exists a lattice  $M \subset V$  which is invariant under  $\Gamma$ , i.e.,  $M$  is a free abelian subgroup of  $V$  such that  $V = M \otimes \mathbb{R}$  and  $M$  is invariant under  $\Gamma$ . Thus  $M$  is a finitely generated  $\mathbb{Z}[\Gamma]$ -module. It defines a local system  $\mathcal{M}$  of free  $\mathbb{Z}$ -modules on  $X$ . Set

$$(2.17) \quad C^q(K; M) = C^q(\tilde{K}; \mathbb{Z}) \otimes_{\mathbb{Z}[\Gamma]} M, \quad q = 0, \dots, n,$$

and let  $H^*(K; M) = H^*(X; M)$  denote the cohomology of the corresponding complex. Now  $C^*(K; M)$  is a complex of finitely generated free  $\mathbb{Z}$ -modules and we have

$$C^*(K; V) = C^*(K; M) \otimes \mathbb{R}.$$

So Lemma 2.2 applies to this situation. As above, we may also consider the set  $C^q(K, \mathcal{M})$  of  $\mathcal{M}$ -valued  $q$ -cochains [Wh, Chapt. VI]. The complex  $C^*(K; \mathcal{M})$  computes the cohomology  $H^*(K; \mathcal{M}) = H^*(X; \mathcal{M})$  and we have a canonical isomorphism

$$H^*(X, M) \cong H^*(X, \mathcal{M}).$$

Each  $H^q(X, \mathcal{M})$  is a finitely generated  $\mathbb{Z}$ -module. Let  $H^q(X, \mathcal{M})_{\text{tors}}$  be the torsion subgroup and

$$H^q(X; \mathcal{M})_{\text{free}} = H^q(X, \mathcal{M}) / H^q(X, \mathcal{M})_{\text{tors}}.$$

We identify  $H^q(X, \mathcal{M})_{\text{free}}$  with a subgroup of  $H^q(X, E)$ . Let  $\langle \cdot, \cdot \rangle_q$  be the inner product in  $H^q(X, E)$  induced by the  $L^2$ -metric on  $\mathcal{H}^q(X, E)$ . Let  $e_1, \dots, e_{r_q}$  be a basis of  $H^q(X, \mathcal{M})_{\text{free}}$  and let  $G_q$  be the Gram matrix with entries  $\langle e_k, e_l \rangle$ . Put

$$R_q(\chi, h) = \sqrt{|\det G_q|}, \quad q = 0, \dots, n.$$

Define the “regulator”  $R(\chi, h)$  by

$$R(\chi, h) = \prod_{q=0}^n R_q(\chi, h)^{(-1)^q}.$$

**Proposition 2.3.** *Let  $\chi$  be a unimodular representation of  $\Gamma$  on a finite-dimensional  $\mathbb{R}$ -vector space  $E$ . Let  $M \subset E$  be a  $\Gamma$ -invariant lattice and let  $\mathcal{M}$  be the associated local system of finitely generated free  $\mathbb{Z}$ -modules on  $X$ . Let  $h$  be a fibre metric in the flat vector bundle  $E = \mathcal{M} \otimes \mathbb{R}$ . Then we have*

$$\tau_X(\chi, h) = R(\chi, h) \cdot \prod_{q=0}^n |H^q(X, \mathcal{M})_{\text{tors}}|^{(-1)^{q+1}}.$$

We will also consider complex representations. A complex representation has an underlying real representation and we need to understand the relation between the Reidemeister torsion for the two representations. This is answered by the following lemma.

**Lemma 2.4.** *Let  $\chi: \Gamma \rightarrow \mathrm{GL}(V)$  be a unimodular, acyclic representation in a finite-dimensional complex vector space  $V$ . Let  $\chi^{\mathbb{R}}$  be the corresponding real representation in  $V^{\mathbb{R}}$ . Then we have*

$$\tau_X(\rho^{\mathbb{R}}) = \tau_X(\rho)^2.$$

*Proof.* The  $p$ -simplexes of  $K$  form a basis of  $C^p(\tilde{K}; \mathbb{Z})$  as a  $\mathbb{Z}[\Gamma]$  module. With respect to these bases, the coboundary operator  $\tilde{d}_p: C^p(\tilde{K}; \mathbb{Z}) \rightarrow C^{p+1}(\tilde{K}; \mathbb{Z})$  is given by a matrix  $(a_{ij})$  with  $a_{ij} \in \mathbb{Z}[\Gamma]$ . Let  $\chi: \mathbb{Z}[\Gamma] \rightarrow \mathrm{End}(V)$  be defined by

$$\chi\left(\sum_j n_j \gamma_j\right) = \sum_j n_j \chi(\gamma_j), \quad n_j \in \mathbb{Z}, \gamma_j \in \Gamma.$$

It follows that the coboundary operator  $d_p^X: C^p(K; V) \rightarrow C^{p+1}(K; V)$  is given by the matrix  $(\chi(a_{ij}))$ . Choose an inner product in  $V$ . Let

$$\Delta_p^X = (d_p^X)^* \circ d_p^X + d_{p-1}^X \circ (d_{p-1}^X)^*.$$

Since  $\chi$  is acyclic,  $\Delta_p^X$  is invertible. By Lemma 2.1 the Reidemeister torsion is given by

$$\tau_X(\chi) = \prod_{j=1}^n (\det \Delta_j^X)^{(-1)^{j+1} j/2}.$$

A similar formula holds for  $\tau_X(\chi^{\mathbb{R}})$ . Thus in order to prove the lemma it suffices to prove that  $\det \Delta_j^{\chi^{\mathbb{R}}} = (\det \Delta_j^X)^2$ ,  $j = 1, \dots, n$ . For a linear operator  $A: W \rightarrow W$  in a complex vector space  $W$  let  $A^{\mathbb{R}}$  denote the corresponding real operator in  $W^{\mathbb{R}}$ . By construction we have  $\Delta_j^{\chi^{\mathbb{R}}} = (\Delta_j^X)^{\mathbb{R}}$ . Thus it suffices to prove that for any linear operator  $A$  in a complex vector space we have  $\det A^{\mathbb{R}} = |\det A|^2$ . Using the Jordan normal form of  $A$ , the problem is reduced to the one-dimensional case. Let  $A: \mathbb{C} \rightarrow \mathbb{C}$  be the multiplication by  $z = u + iv$ . Then  $A^{\mathbb{R}} = \begin{pmatrix} u & -v \\ v & u \end{pmatrix}$  and therefore  $\det A^{\mathbb{R}} = u^2 + v^2 = |z|^2 = |\det A|^2$  which proves the lemma.  $\square$

### 3. ARITHMETIC GROUPS

In this section we recall some facts about quaternion algebras and arithmetic subgroups of  $\mathrm{SL}(2, \mathbb{C})$  derived from quaternion algebras.

Let  $F$  be a field of characteristic zero. Every quaternion algebra over  $F$  can be described as 4-dimensional  $F$ -vector space with basis  $1, i, j, k$  satisfying the following relations:

$$i^2 = a, \quad j^2 = b, \quad ij = k, \quad ji = -k,$$

where  $a, b \in F^\times$ . This algebra will be denoted by  $H(a, b; F)$  or simply  $H(a, b)$ . Let  $x \in H \mapsto \bar{x} \in H$  be the involution defined by

$$\overline{x_0 + x_1 i + x_2 j + x_3 k} = x_0 - x_1 i - x_2 j - x_3 k.$$

Then the norm and the trace in  $H$  are defined by

$$N(x) = x\bar{x}, \quad \text{Tr}(x) = x + \bar{x}, \quad x \in H.$$

It follows from Wedderburn's structure theorem for simple algebras that a quaternion algebra  $H$  over  $F$  is either a division algebra or is isomorphic to  $M_2(F)$  (see [Lam, Theorem II.2.7], [MR, Theorem 2.1.7]). In the latter case  $H$  is called split over  $F$ .

Let  $D$  be a division algebra over  $F$  of degree  $d^2$ ,  $d \leq 2$ . Let  $m = 2/d$ . There is an associated almost simple semisimple algebraic group  $\text{SL}_m(D)$ . It is defined as the functor from  $F$ -algebras to groups which assigns to an  $F$ -algebra  $R$  the group

$$\text{SL}_m(D)(R) := \{x \in M_m(D \otimes_F R) : N(x) = 1\}.$$

The algebraic groups  $\text{SL}_m(D)$  are the forms of  $\text{SL}_2$  over  $F$  [Min, Theorem 27.9]. If  $d = 1$ , we have  $D = F$  and  $\text{SL}_2(D) = \text{SL}_2$ . If  $d = 2$ ,  $D$  is a quaternion division algebra.

We consider  $\text{SL}_2$  as algebraic group over  $F$ . For  $n \in \mathbb{N}$  let  $V(n) := S^n(F^2)$  be the  $n$ -th symmetric power of  $F^2$  and let  $\text{Sym}^n : \text{SL}_2 \rightarrow \text{GL}(V(n))$  be the  $n$ -th symmetric power of the standard representation  $\rho : \text{SL}_2 \rightarrow \text{GL}_2$  of  $\text{SL}_2$ . The following lemma is a special case of [Tit, Theorem 3.3]. For the convenience of the reader we include the proof which was kindly communicated to us by Skip Garibaldi.

**Lemma 3.1.** *Let  $G'$  be a form of  $\text{SL}_2/F$ . For every even  $n$  there exists a  $F$ -rational representation of  $G'$  on  $V(n)$  which is equivalent to  $\text{Sym}^n$  over  $\overline{F}$ .*

*Proof.* First we observe that the Dynkin diagram  $A_1$  has no nontrivial automorphisms. Therefore  $\text{SL}_2$  has no outer automorphisms and

$$(3.18) \quad \text{Aut}(\text{SL}_2) = \text{PGL}_2.$$

Hence the forms of  $\text{SL}_2$  are classified by the Galois cohomology set  $H^1(\overline{F}/F, \text{PGL}_2)$  (see [Min, pp. 181]). Let  $\mu : \text{SL}_2(\overline{F}) \rightarrow G'(\overline{F})$  be an isomorphism (which exists by assumption). For  $\sigma \in \text{Gal}(\overline{F}/F)$  put

$$(3.19) \quad \varphi_\sigma = \mu^{-1} \circ \sigma \circ \mu \circ \sigma^{-1}.$$

Then  $\varphi_\sigma$  is an automorphism of  $\text{SL}_2/\overline{F}$ . By (3.18) there exists  $a_\sigma \in \text{PGL}_2(\overline{F})$  such that  $\varphi_\sigma = \text{Int}(a_\sigma)$ . As in [Min, Theorem 26.10] it follows that  $\sigma \in \text{Gal}(\overline{F}/F) \mapsto a_\sigma \in \text{PGL}_2(\overline{F})$  is the cocycle that represents  $G'$ . Let  $a'_\sigma \in \text{SL}_2(\overline{F})$  be any lift of  $a_\sigma$ . Let  $n \in \mathbb{N}$  be even. Then  $\text{Sym}^n$  is trivial on the center of  $\text{SL}_2$  and therefore the map

$$\sigma \in \text{Gal}(\overline{F}/F) \mapsto \text{Sym}^n(a'_\sigma) \in \text{GL}(V(n) \otimes \overline{F})$$

is a well-defined cocycle. By Hilbert's theorem 90 we have  $H^1(\overline{F}/F, \text{GL}_N) = 1$  for every  $N$ . Hence there exists  $x \in \text{GL}(V(n) \otimes \overline{F})$  such that

$$(3.20) \quad \text{Sym}^n(a'_\sigma) = x^{-1} \sigma(x), \quad \forall \sigma \in \text{Gal}(\overline{F}/F).$$

Define the map  $\rho_n : G' \rightarrow \text{GL}(V(n))$  by

$$(3.21) \quad \rho_n := \text{Int}(x) \circ \text{Sym}^n \circ \mu.$$

This is a representation of  $G'$  over  $\overline{F}$  which over  $\overline{F}$  is equivalent to  $\text{Sym}^n$ . By (3.19) the action of  $\text{Gal}(\overline{F}/F)$  on  $G'(\overline{F})$  is given by

$$(3.22) \quad \mu(\sigma(g)) = \text{Int}(a_\sigma) \cdot \sigma(\mu(g)), \quad g \in G'(\overline{F}), \sigma \in \text{Gal}(\overline{F}/F).$$

Using (3.20), (3.21) and (3.22), it follows that for all  $\sigma \in \text{Gal}(\overline{F}/F)$  and  $g \in G'(\overline{F})$  we get

$$\begin{aligned} \rho_n(\sigma(g)) &= \text{Int}(x) \text{Sym}^n(\text{Int}(a'_\sigma)\sigma(\mu(g))) = \text{Int}(x \text{Sym}^n(a'_\sigma))\sigma(\text{Sym}^n(\mu(g))) \\ &= \text{Int}(\sigma(x))\sigma(\text{Sym}^n(\mu(g))) = \sigma(\text{Int}(x) \text{Sym}^n(\mu(g))) = \sigma(\rho_n(g)). \end{aligned}$$

Thus  $\rho_n$  commutes with the action of  $\text{Gal}(\overline{F}/F)$  and hence is defined over  $F$ .  $\square$

Now let  $F$  be an imaginary quadratic number field and let  $\mathcal{O}_F$  be the ring of integers of  $F$ . Fix an embedding  $F \subset \mathbb{C}$ . As explained above, every quaternion algebra over  $F$ , which is not isomorphic to  $M_2(F)$ , is a division algebra. This is the case which is of interest for us. So let  $D$  be a quaternion division algebra over  $F$ . Put

$$D^1 = \{x \in D : N(x) = 1\}, \quad D^0 = \{x \in D : \text{Tr}(x) = 0\}.$$

Let

$$G = R_{F/\mathbb{Q}} \text{SL}_1(D)$$

be the algebraic group obtained from  $\text{SL}_1(D)$  by restriction of scalars. Then  $G$  is defined over  $\mathbb{Q}$ . We have

$$(3.23) \quad G(\mathbb{Q}) \cong D^1, \quad G(\mathbb{R}) \cong \text{SL}_1(D)(F \otimes_{\mathbb{Q}} \mathbb{R}) = (D \otimes_F \mathbb{C})^1$$

and there is an isomorphism of  $\mathbb{C}$ -algebras

$$(3.24) \quad \varphi : (D \otimes_F \mathbb{C})^1 \xrightarrow{\cong} \text{SL}(2, \mathbb{C}).$$

Let  $\mathfrak{o}$  be an order in  $D$ . Recall that this means that  $\mathfrak{o}$  is a finitely generated  $\mathcal{O}_F$ -module which contains an  $F$ -basis of  $D$  and which is also a subring of  $D$ . Let

$$\mathfrak{o}^1 = \{x \in \mathfrak{o} : N(x) = 1\}, \quad \mathfrak{o}^0 = \{x \in \mathfrak{o} : \text{Tr}(x) = 0\}.$$

Then  $\mathfrak{o}^1 \subset D^1$  and  $\mathfrak{o}^0 \subset D^0$  is a lattice which is invariant under  $\mathfrak{o}^1$  with respect to the adjoint action of  $\text{SL}_1(D)$  on  $D^0$ . Thus by (3.23) it follows that  $\mathfrak{o}^1$  corresponds to an arithmetic subgroup  $\Gamma^1 \subset G(\mathbb{Q})$ . Put

$$\Gamma = \varphi(\mathfrak{o}^1)$$

Then  $\Gamma$  is a discrete subgroup of  $\text{SL}(2, \mathbb{C})$ . Such a group is called an arithmetic subgroup derived from a quaternion algebra. The following lemma is a consequence of [EGM, Theorem 10.1.2].

**Lemma 3.2.** *Let  $\Gamma \subset \text{SL}(2, \mathbb{C})$  be derived from a quaternion division algebra over  $F$ . Then the quotient  $\Gamma \backslash \text{SL}(2, \mathbb{C})$  is compact.*

By imposing further conditions on  $D$ , one can achieve that  $\Gamma$  is torsion free [EGM, Theorem 10.1.2]. Alternatively, we can pass to a normal subgroup of  $\Gamma$  of finite index which is torsion free.

Let  $V_1(n) = R_{F/\mathbb{Q}}(V(n))$ . Then  $V_1(n)$  is a  $\mathbb{Q}$ -vector space and it follows from Lemma 3.1 that for every even  $n$  there exist a  $\mathbb{Q}$ -rational representation

$$\rho_n: G \rightarrow \mathrm{GL}(V_1(n))$$

which over  $\mathbb{R}$  is equivalent to  $\mathrm{Sym}^n$  of  $\mathrm{SL}_2$ . Since  $\Gamma^1$  is an arithmetic subgroup of  $G(\mathbb{Q})$ , there exists a lattice  $L \subset V_1(n)$  which is stable under  $\Gamma^1$  (see [Min, Proposition 28.9]). Using the isomorphism (3.24) this implies the following proposition.

**Proposition 3.3.** *Let  $\Gamma \subset \mathrm{SL}(2, \mathbb{C})$  be an arithmetic subgroup derived from a quaternion algebra over an imaginary quadratic number field. Then for each even  $n \in \mathbb{N}$  there exists a lattice  $M_n \subset S^n(\mathbb{C}^2)$  which is stable under  $\Gamma$  with respect to  $\mathrm{Sym}^n$ .*

We add some remarks about the classification of quaternion algebras over a number field  $F$ . For details see [Vig, Chap. III]. Let  $H$  be a quaternion algebra over  $F$ . Given a place  $v$  of  $F$ , let

$$H_v = H \otimes_F F_v.$$

Over  $\mathbb{C}$  every quaternion algebra splits, i.e., there is an  $\mathbb{C}$ -algebra isomorphism

$$\varphi: H \otimes_F \mathbb{C} \xrightarrow{\cong} M_2(\mathbb{C}).$$

By the Frobenius theorem, a quaternion algebra over  $\mathbb{R}$  is either split or isomorphic to the Hamiltonian quaternions. If  $\mathfrak{p}$  is a finite place of  $F$ , then, up to isomorphism, there is a unique quaternion division algebra over  $F_{\mathfrak{p}}$  [Vig, Theorem II.1.1].  $H$  is called ramified at a given place  $v$ , if  $H_v$  is a division algebra. If  $H = H(a, b; F)$ ,  $a, b \in F^\times$ , the behavior at  $v$  is determined by the Hilbert symbol  $(a, b)_v \in \{\pm 1\}$ .  $H_v$  is split iff  $(a, b)_v = 1$ . It follows from the Hasse-Minkowski principle that  $H$  splits over  $F$  iff  $H_v$  splits for every place  $v$  of  $F$  [MR, Theorem 2.9.6]. For  $H = H(a, b; F)$  this is equivalent to  $(a, b)_v = 1$  for all places  $v$ . Furthermore the number of places where  $H$  is ramified is even. Denote by  $V_f(F)$  and  $V_{\mathbb{R}}(F)$  the set of finite places and real places of  $F$ , respectively. Then for every finite set  $S \subset V_f(F) \cup V_{\mathbb{R}}(F)$  of even cardinality there is a quaternion algebra  $H$  over  $F$  which is unique up to  $F$ -algebra isomorphism, such that the set of places where  $H$  is ramified is equal to  $S$ .

Thus for an imaginary quadratic number field  $F$ , we can pick any nonempty finite set  $S$  of finite places  $\mathfrak{p}$  with  $|S|$  even. Then up to isomorphism, there is a unique quaternion division algebra  $D$  over  $F$  which is ramified exactly at the places  $\mathfrak{p} \in S$ .

#### 4. BOUNDS FOR TORSION IN COHOMOLOGY

We now prove (1.2) of theorem 1.1, which states that the contribution of the terms with cohomological degree 1 and 3 to the alternating sum (1.3) is small. To begin with the  $H^3$  term, it is required to show that

$$(4.25) \quad \log |H^3(X, \mathcal{M}_{2k})| \ll k \log k$$

uniformly over all choices of lattice  $M_{2k}$ . We first apply the isomorphism  $H^3(X, \mathcal{M}_{2k}) \simeq (M_{2k})_\Gamma$ , which follows from computing  $H^3$  using a triangulation of  $X$  and then observing that this is equivalent to computing  $H_0(\Gamma, M_{2k})$  using the dual triangulation. We shall then bound  $(M_{2k})_\Gamma$  by working locally. Let  $V(2k) = S^{2k}(F^2)$ . For each prime  $\mathfrak{p}$  of  $F$ , let  $V_{\mathfrak{p}}(2k)$  be the completion of  $V(2k)$  at  $\mathfrak{p}$ , and let  $M_{2k,\mathfrak{p}}$  be the completion of the image of  $M_{2k}$ .  $M_{2k,\mathfrak{p}}$  is a  $\Gamma$ -stable lattice, and we have

$$\log |(M_{2k})_\Gamma| = \sum_{\mathfrak{p}} \log |(M_{2k,\mathfrak{p}})_\Gamma|.$$

We divide the primes of  $F$  into two sets, which we call unramified and ramified. The first set contains those at which the division algebra  $D$  is unramified and the closure of  $\Gamma$  in  $D_{\mathfrak{p}}^1 \simeq \mathrm{SL}_2(F_{\mathfrak{p}})$  is isomorphic to the standard maximal compact, and the second contains the remainder. The following lemma implies that the unramified primes whose residue characteristic is greater than  $2k$  make no contribution to the sum.

**Lemma 4.1.** *If  $\mathbb{F}_q$  is the field with  $q = p^j$  elements, then the  $d$ -th symmetric power representation of  $\mathrm{SL}_2(\mathbb{F}_q)$  over  $\mathbb{F}_q$  is irreducible for  $d < p$ .*

*Proof.* Denote this representation by  $(\rho, V_d)$ , and let  $N$  and  $\overline{N}$  be nontrivial upper and lower unipotent elements of  $\mathrm{SL}_2(\mathbb{F}_q)$ . When  $\rho(N) - I$  and  $\rho(\overline{N}) - I$  are expressed in the standard monomial basis of  $V_d$  they are strictly upper (resp. lower) triangular, and all their entries in the spaces lying immediately above (resp. below) the diagonal are nonzero. It follows that the only subspaces of  $V_d$  which are invariant under both of these operators are  $\{0\}$  and  $V_d$ .  $\square$

Because  $F$  was imaginary quadratic, lemma 4.1 allows us to assume that  $N\mathfrak{p} \leq 4k^2$ . The following proposition gives a bound for  $|(M_{2k,\mathfrak{p}})_\Gamma|$  at unramified primes which is uniform in  $\mathfrak{p}$  and  $M_{2k}$ , and gives us the required bound (4.25) when summed over those  $\mathfrak{p}$  with  $N\mathfrak{p} \leq 4k^2$ . For ramified primes we shall adapt the proof of the proposition to give bounds which are only uniform in the lattice, but this will be sufficient for our purposes as the number of such primes is bounded independently of  $k$ .

**Proposition 4.2.** *Let  $K$  be a  $p$ -adic field,  $\mathcal{O} \subset K$  the ring of integers in  $K$ ,  $\varpi$  a uniformiser of  $\mathcal{O}$ , and  $q = |\mathcal{O}/\varpi|$ . Consider the  $2k$ -th symmetric power representation of  $G = \mathrm{SL}(2, \mathcal{O})$  on a vector space  $V$  of dimension  $2k + 1$ . If  $L \subset V$  is any  $G$ -stable lattice and  $L/L'$  is the largest  $G$ -invariant quotient of  $L$ , then we have*

$$\log_q |L : L'| \ll k/q$$

where the implied constant is absolute, i.e. independent of  $L$  and  $K$ .

*Proof.* The proof proceeds in two steps. First, choose  $a \in \mathcal{O}^\times$  to have the highest possible order in all the quotient groups  $\mathcal{O}^\times/(1 + \varpi^t \mathcal{O})$ , or equivalently so that  $a$  is a primitive root in  $(\mathcal{O}/\varpi)^\times$  and  $a^{q-1} \notin 1 + \varpi^2 \mathcal{O}$ . Define  $T$  to be the diagonal matrix

$$T = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}.$$

We know that  $(T - 1)L \subset L'$ , so it would suffice to bound the order of  $L/(T - 1)L$ . This is infinite, however, because  $T$  fixes the vector  $(xy)^k$ . Instead, we shall prove that if  $V_0$  is the space spanned by  $(xy)^k$  and we define  $J$  and  $J'$  to be the intersections of  $V_0$  with  $L$  and  $L'$ , we have

$$(4.26) \quad \log_q |L : L'| \ll \frac{k}{q} + \log_q |J : J'|.$$

The second step is to bound  $|J : J'|$ , for which we shall need to consider the action of the unipotents on  $V$ .

We carry out the first step by decomposing  $V$  into a direct sum of subspaces  $\mathcal{V}_i$ , for  $i \in \mathbb{Z}_{\geq 0}$  and  $i = \infty$ . For  $i \in \mathbb{Z}_{\geq 0}$ , we define  $\mathcal{V}_i$  to be the span of the eigenspaces of  $T - 1$  with eigenvalue  $\lambda$  satisfying  $\text{ord}_\varpi \lambda = i$ , and define  $\mathcal{V}_\infty = \text{span}\{(xy)^k\}$ . In other words, if  $V_t$  is the span of the monomial  $x^{k+t}y^{k-t}$ , we define

$$\begin{aligned} \mathcal{V}_0 &= \bigoplus_{2t \neq 0 \pmod{q-1}} V_t \\ \mathcal{V}_i &= \bigoplus_{\substack{2t=j \pmod{(q-1)p^{i-1}} \\ j \neq 0 \pmod{p}}} V_t \\ \mathcal{V}_\infty &= V_0. \end{aligned}$$

Note that for  $i \geq 1$  we have the bound

$$\dim \mathcal{V}_i \leq Ck/qp^{i-1}$$

for some absolute constant  $C$ . Let  $\Pi_i$  be the projection onto  $\mathcal{V}_i$ , and define  $\mathcal{V}'_i$  by

$$\mathcal{V}'_i = \bigoplus_{\substack{j > i, \\ j = \infty}} \mathcal{V}_j$$

so that  $\mathcal{V}'_i$  is the span of the eigenspaces of  $T - 1$  with eigenvalue  $\lambda$  satisfying  $\text{ord}_\mathfrak{p} \lambda > i$ . Let  $L_i = L \cap \mathcal{V}'_i$  and  $L'_i = L' \cap \mathcal{V}'_i$ . We shall prove (4.26) using induction on  $L_i$ , by establishing the inequalities

$$(4.27) \quad |L : L'| \leq |L_0 : L'_0|,$$

$$(4.28) \quad \log_q |L_i : L'_i| \leq Ck(i+1)/qp^i + \log_q |L_{i+1} : L'_{i+1}|$$

for all  $i \geq 0$ .

To prove (4.27), note that because  $T$  commutes with all co-ordinate projections we have

$$(T - 1)\Pi_0 L \subseteq \Pi_0 L' \subseteq \Pi_0 L,$$

and that all the above inclusions must be equalities because the determinant of  $T - 1$  restricted to  $\mathcal{V}_0$  is a unit of  $\mathcal{O}$ . To deduce (4.27) from the equality  $\Pi_0 L = \Pi_0 L'$ , let  $\{v_i\}$  be a system of coset representatives for  $L_0/L'_0$ , and let  $x \in L$  be given. Because  $\Pi_0 L = \Pi_0 L'$ , there exists  $y \in L'$  such that  $\Pi_0 y = \Pi_0 x$ , and so  $x - y \in L_0$ . Therefore there exists  $v_i$  such that  $x - y - v_i \in L'_0$ , which implies  $x \in L' + v_i$ .

Applying the same argument to the lattices  $L_i$  and  $L'_i$  gives

$$(T - 1)\Pi_{i+1} L_i \subseteq \Pi_{i+1} L'_i \subseteq \Pi_{i+1} L_i,$$

and on taking determinants we have

$$\log_q |\Pi_{i+1} L_i : \Pi_{i+1} L'_i| \leq \log_q |\Pi_{i+1} L_i : (T - 1)\Pi_{i+1} L_i| \leq (i + 1) \dim \mathcal{V}_{i+1} \leq Ck(i + 1)/qp^i,$$

where  $i + 1$  appears because it is the  $\varpi$ -adic valuation of the eigenvalues of  $T - 1$  on  $\mathcal{V}_{i+1}$ . Repeating the above argument on coset representatives, we see that this implies (4.28). On summing (4.27) and (4.28) we obtain

$$\begin{aligned} \log_q |L : L'| &\ll \sum_{t=1}^i \frac{kt}{qp^{t-1}} + \log_q |L_i : L'_i| \\ &\ll \frac{k}{q} + \log_q |L_i : L'_i|, \end{aligned}$$

from which (4.26) follows by letting  $i \rightarrow \infty$ .

To bound  $|J : J'|$ , we may assume that  $J$  is generated by  $(xy)^k$ , and act on this monomial by an upper triangular element  $N \in \mathrm{SL}(2, \mathcal{O})$ .  $(N - 1)(xy)^k$  will then contain a nonzero term of the form  $x^{k+1}y^{k-1}$ , and using the endomorphism algebra generated by  $T$  we may in fact show that  $L'$  contains a monomial  $\varpi^s x^{k+1}y^{k-1}$  with  $s$  small. Repeating this argument with a lower triangular matrix  $\overline{N}$  gives the required bound on  $|J : J'|$ . The statement we require about the endomorphism algebra generated by  $T$  is the following:

**Lemma 4.3.** *For  $\gamma \in \mathcal{O}^\times$ , let  $E$  be the ring of endomorphisms of  $\mathcal{O}^{2k+1}$  generated over  $\mathcal{O}$  by the diagonal matrix  $A$  with entries  $1, \gamma, \dots, \gamma^{2k}$ . Then for  $0 \leq j \leq 2k$ ,  $E$  contains  $p_j(\gamma^j)\pi_j$  where  $\pi_j$  is the projection onto the  $j$ th co-ordinate and  $p_j(z)$  is the polynomial*



$$p_j(z) = \prod_{\substack{0 \leq i \leq 2k, \\ i \neq j}} (z - \gamma^i).$$

*Proof.* This follows easily by considering the matrix  $p_j(A)$ , whose only nonzero entry is  $p_j(\gamma^j)$  in the  $(j, j)$ th co-ordinate. □

Consider the polynomial  $x^k(y+x)^k - (xy)^k \in L'$ . Applying the lemma with the choice of  $\gamma = a^2$  and the projection onto the space spanned by  $x^{k+1}y^{k-1}$ , we see that  $L'$  contains

$$k \prod_{\substack{-2k-2 \leq 2i \leq 2k-2, \\ i \neq 0}} (1 - a^{2i}) x^{k+1} y^{k-1} = u \varpi^\alpha x^{k+1} y^{k-1},$$

where  $u$  is a unit of  $\mathcal{O}$  and  $\alpha \in \mathbb{Z}$ . In the product above, at most  $Ck/q$  terms are divisible by  $\varpi$ , at most  $Ck/qp$  are divisible by  $\varpi^2$  and so on, so that we may bound  $\alpha$  by

$$\begin{aligned} \alpha &\ll \log_q k + \frac{k}{q} + \frac{k}{qp} + \dots \\ &\ll \frac{k}{q}. \end{aligned}$$

By applying the same argument to the monomial  $\varpi^\alpha x^{k+1} y^{k-1}$ , but now with an element of the opposite unipotent, we see that  $\varpi^\beta (xy)^k \in L'$  with  $\beta \ll k/q$ . This gives the required bound on  $|J : J'|$ , and completes the proof. □

Using proposition 4.2, we see that the contribution to the order of  $\log |(M_{2k})_\Gamma|$  from unramified primes is at most a constant times

$$\sum_{N\mathfrak{p} \leq 4k^2} \frac{k \log N\mathfrak{p}}{N\mathfrak{p}} \ll k \log k.$$

As the number of ramified primes is bounded independently of  $k$ , in the remaining cases it suffices to prove bounds of the form  $\log |(M_{2k,\mathfrak{p}})_\Gamma| \ll k$  where the constant is allowed to depend on  $\mathfrak{p}$ .

First, consider a prime  $\mathfrak{p}$  at which  $D$  is split but the closure of  $\Gamma$  in  $D_\mathfrak{p}^1 \simeq \mathrm{SL}_2(F_\mathfrak{p})$  is not isomorphic to  $\mathrm{SL}(2, \mathcal{O})$ , and denote this closure by  $G$ . As in the unramified case, let  $V$  be the  $2k$ -th degree symmetric power representation of  $\mathrm{SL}_2(F_\mathfrak{p})$ , let  $L \subset V$  be any  $G$ -stable lattice, and  $L/L'$  its largest  $G$ -invariant quotient. As  $G$  is open, we know it contains an element

$$T = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}$$

for some  $a \in \mathcal{O}^\times$ . As in the unramified case, define  $\mathcal{V}_i$  to be the sum of the eigenspaces of  $T - 1$  with eigenvalue  $\lambda$  satisfying  $\text{ord}_\varpi \lambda = i$ , let  $\mathcal{V}'_i$  be the sum of the eigenspaces with  $\text{ord}_\varpi \lambda > i$ , and let  $J = L \cap \mathcal{V}_\infty$  and  $J' = L' \cap \mathcal{V}_\infty$ . Because we have inequalities

$$\dim \mathcal{V}_i \leq C(a) \frac{k}{p^i}$$

uniformly in  $i$  and  $k$ , we may use these subspaces to perform the same reduction argument which bounds  $|L : L'|$  in terms of  $|J : J'|$  to obtain

$$\log_q |L : L'| \leq \log_q |J : J'| + O_a(k).$$

$G$  also contains upper and lower unipotent elements

$$N = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \overline{N} = \begin{pmatrix} 1 & 0 \\ n & 1 \end{pmatrix}$$

for some  $n \in \mathcal{O}$ , and arguing as in the unramified case we may use these to show that

$$\begin{aligned} \log_q |J : J'| &\ll_n \log_q k + \sum_{\substack{-2k-2 \leq 2i \leq 2k+2, \\ i \neq 0}} \text{ord}_\varpi(1 - a^{2i}) \\ &\ll_{n,a} \log_q k + \sum_{i \geq 0} \frac{k}{p^i} \\ &\ll_{n,a} k. \end{aligned}$$

We therefore have an upper bound of  $\log_q |L : L'| \ll k$ , where the constant may depend on  $\mathfrak{p}$ .

Finally, suppose that  $D$  is ramified at  $\mathfrak{p}$ . We continue to denote the closure of  $\Gamma$  in  $D_{\mathfrak{p}}^1$  by  $G$ . Using the adjoint representation, we may realise  $D_{\mathfrak{p}}^1$  as an algebraic subgroup of  $GL_3(F_{\mathfrak{p}})$ . There is an open subgroup  $U$  of  $GL_3(F_{\mathfrak{p}})$  such that the power series  $\log(1 + A)$  converges  $p$ -adically for  $1 + A \in U$ , from which we see that there is an open subgroup  $U' \subset D_{\mathfrak{p}}^1$  on which we may define an inverse exponential map. By applying this map to  $U' \cap G$  we obtain a lattice  $N$  in the Lie algebra of  $D_{\mathfrak{p}}^1$  which annihilates  $L/L'$ . We may then extend scalars to a field over which  $D_{\mathfrak{p}}$  splits so that we are again dealing with  $SL_2$ , and apply the method of proposition 4.2 to obtain an upper bound of  $\log_q |L : L'| \ll k$ . This completes the proof of (4.25).

In the case of  $H^1$ , we may apply the long exact sequence in cohomology associated to the short exact sequence

$$0 \longrightarrow M_{2k} \xrightarrow{\times d} M_{2k} \longrightarrow M_{2k}/d \longrightarrow 0,$$

of which one segment is

$$H^0(\Gamma, M_{2k}/d) \longrightarrow H^1(\Gamma, M_{2k}) \xrightarrow{\times d} H^1(\Gamma, M_{2k}).$$

Therefore the torsion in  $H^1(\Gamma, M_{2k})$  is bounded by the limit of the  $\Gamma$ -invariants in  $M_{2k}/d$  as  $d$  becomes divisible by arbitrarily high powers of every prime. We may again bound this by working locally, and so wish to bound the order of the module of  $\Gamma$ -invariants  $(M_{2k,\mathfrak{p}}/\varpi^t M_{2k,\mathfrak{p}})^\Gamma$  for each  $\mathfrak{p}$  and arbitrary  $t$ . If we let  $L$  be the inverse image of  $(M_{2k,\mathfrak{p}}/\varpi^t M_{2k,\mathfrak{p}})^\Gamma$  in  $V_{\mathfrak{p}}(2k)$ , we see that  $\Gamma$  acts trivially on  $L/\varpi^t M_{2k,\mathfrak{p}}$ . We may therefore apply our bounds for  $L_\Gamma$  to obtain

$$\log |H^1(\Gamma, M_{2k})| \ll k \log k$$

as required.

## 5. PROOF OF THE MAIN RESULTS

Let  $X = \Gamma \backslash \mathbb{H}^3$  be a compact oriented hyperbolic 3-manifold defined by a cocompact torsion free discrete subgroup  $\Gamma \subset \mathrm{SL}(2, \mathbb{C})$ . For  $m \in \mathbb{N}$  let  $\rho_m: \mathrm{SL}(2, \mathbb{C}) \rightarrow \mathrm{GL}(S^m(\mathbb{C}^2))$  be the  $m$ -th symmetric power of the standard representation of  $\mathrm{SL}(2, \mathbb{C})$ . We regard  $S^m(\mathbb{C}^2)$  as an  $\mathbb{R}$ -vector space. Let  $\rho_m^\mathbb{R}$  be the corresponding real representation. Let  $E_m \rightarrow X$  be the flat vector bundle associated to  $\rho_m$  or  $\rho_m^\mathbb{R}$ , respectively. By [BW, Chapt. VII, Theorem 6.7] we have  $H^*(X, E_m) = 0$ . Therefore, the Reidemeister torsions  $\tau_X(\rho_m)$  and  $\tau_X(\rho_m^\mathbb{R})$  of  $X$  with respect to the restrictions to  $\Gamma$  of  $\rho_m$  and  $\rho_m^\mathbb{R}$ , respectively, are well defined. By Lemma 2.4 we have

$$(5.29) \quad \tau_X(\rho_m^\mathbb{R}) = \tau_X(\rho_m)^2.$$

Then it follows from [Mu2, Corollary 1.2] that

$$(5.30) \quad -\log \tau_X(\rho_m^\mathbb{R}) = \frac{\mathrm{vol}(X)}{2\pi} m^2 + O(m)$$

as  $m \rightarrow \infty$ .

Now let  $\Gamma \subset \mathrm{SL}(2, \mathbb{C})$  be an arithmetic subgroup derived from a quaternion division algebra over an imaginary quadratic number field  $F$ . By Lemma 3.2 it is cocompact. By passing to a normal subgroup of finite index we can assume that  $\Gamma$  is torsion free. Let  $m \in \mathbb{N}$  be even. By Proposition 3.3 there exists a lattice  $M_m \subset S^m(\mathbb{C}^2)$  which is stable under  $\Gamma$ . Let  $\mathcal{M}_m$  be the associated local system of free  $\mathbb{Z}$ -modules over  $X$ . Let  $\mathcal{M}_m(\mathbb{R}) := \mathcal{M}_m \otimes \mathbb{R}$ . This is the local system associated to the restriction of  $\rho_m$  to  $\Gamma$ . Hence  $H^*(X, \mathcal{M}_m(\mathbb{R})) \cong H^*(X, E_m)$  and it follows from the remark above that  $H^*(X, \mathcal{M}_m(\mathbb{R})) = 0$ . Therefore  $H^*(X, \mathcal{M}_n)$  is a finite abelian group. Note that  $H^0(X, \mathcal{M}_n) = M_n^\Gamma = 0$ . Denote by  $|H^p(X, \mathcal{M}_n)|$  the order of  $H^p(X, \mathcal{M}_n)$ ,  $p = 1, 2, 3$ . Then by Proposition 2.3 we get

$$(5.31) \quad \tau_X(\rho_n^\mathbb{R}) = \prod_{p=1}^3 |H^p(X, \mathcal{M}_n)|^{(-1)^{p+1}}.$$

Combining (5.30) and (5.31), we obtain Theorem 1.1.

**Remark:** It follows from (5.31) that the quantity  $\sum_{p=1}^3 (-1)^p \log |H^p(X, \mathcal{M}_{2k})|$  is independent of the choice of lattice  $M_{2k} \subset V(2k)$ , but this may also be deduced in an elementary way from the long exact sequence in cohomology associated to any inclusion of lattices  $M'_{2k} \subset M_{2k}$ . Let  $T = M_{2k}/M'_{2k}$ . The long exact sequence associated to

$$0 \longrightarrow M'_{2k} \longrightarrow M_{2k} \longrightarrow T \longrightarrow 0$$

implies that

$$(5.32) \quad \sum_{p=1}^3 (-1)^p \log |H^p(X, \mathcal{M}_{2k})| - \sum_{p=1}^3 (-1)^p \log |H^p(X, \mathcal{M}'_{2k})| = \sum_{p=1}^3 (-1)^p \log |H^p(X, \mathcal{T})|.$$

The groups  $H^p(X, \mathcal{T})$  are the cohomology groups of a complex  $\{C^p(T)\}$ , where  $C^p(T)$  is the group of  $T$ -valued cochains of degree  $p$  in some triangulation of  $X$ . Because  $X$  is three dimensional, its Euler characteristic  $\chi(X)$  is zero and so we have

$$\sum_{p=1}^3 (-1)^p \log |C^p(T)| = \chi(X) \log |T| = 0.$$

This implies the same relation for the groups  $H^p(X, \mathcal{T})$ , and so (5.32) is zero as required.

The proof of Theorem 1.3 follows from [Mu2, Theorem 1.5]. Let  $\rho: SL(2, \mathbb{C}) \rightarrow GL(V)$  be an irreducible finite-dimensional complex representation of  $SL(2, \mathbb{C})$ , regarded as real Lie group. The Ruelle zeta function  $R_\rho(s)$  considered in [Mu2] is related to the zeta function  $R(s; \rho)$  by

$$R(s; \rho) = R_\rho(s)^{-1}.$$

The restriction of  $\rho$  to  $\Gamma$  defines a flat vector bundle  $E_\rho \rightarrow X$ . By [MM, Lemma 3.1] it carries a canonical fibre metric  $h$ . Let  $H^*(X, E_\rho)$  be the de Rham cohomology of  $E_\rho$ -valued differential forms. Let  $\theta$  be the Cartan involution of  $SL(2, \mathbb{C})$  with respect to  $SU(2)$ . Let  $\rho_\theta := \tau \circ \rho$ . If  $\rho_\theta \not\cong \rho$ , then it follows from [BW, Chapt. VII, Theorem 6.7] that  $H^*(X, E_\rho) = 0$ . Let  $T_X(\rho)$  be the Ray-Singer analytic torsion of  $X$  with respect to the restriction of  $\rho$  to  $\Gamma$ . Note that in order to define the analytic torsion, we need to choose a fibre metric in  $E_\rho$ . However, since  $H^*(X, E_\rho) = 0$  and the dimension of  $X$  is odd, the analytic torsion is independent of the any fibre metric [Mu1, Corollary 2.7], which justifies the notation. By [Mu2, Theorem 1.5, 1)] the Ruelle zeta function  $R(s; \rho)$  is regular at  $s = 0$  and we have

$$(5.33) \quad |R(0; \rho)| = T_X(\rho)^{-2},$$

Furthermore, by [Mu1, Theorem 1], the analytic torsion  $T_X(\rho)$  equals the Reidemeister torsion  $\tau_X(\rho)$  of  $X$  and  $\rho|_\Gamma$ . Let  $\rho^\mathbb{R}$  be the real representation associated to  $\rho$ . Together with Lemma 2.4 we obtain

$$(5.34) \quad |R(0; \rho)| = \tau_X(\rho^\mathbb{R})^{-1}.$$

Now assume that  $\rho_\theta = \rho$ . Then  $H^*(X, E_\rho)$  may be nonzero. It follows from [Mu2, Theorem 1.5, 2)] that the order of  $R(s; \rho)$  at  $s = 0$  is given by

$$(5.35) \quad \text{ord}_{s=0} R(s; \rho) = 2 \sum_{p=0}^3 (-1)^p p \dim H^p(X, E_\rho)$$

and the leading coefficient  $R^*(0; \rho)$  of the Laurent expansion of  $R(s; \rho)$  at  $s = 0$  equals the analytic torsion  $T_X(\rho; h)^{-2}$ , where  $h$  is the canonical fibre metric  $h$  on  $E_\rho$ . Let  $\tau_X(\rho; h)$  be the Reidemeister torsion with respect to the  $L^2$ -inner product in  $H^*(X, E_\rho)$  defined by the isomorphism with the space  $\mathcal{H}^*(X, E_\rho)$  of  $E_\rho$ -valued harmonic forms. Using again [Mu1, Theorem 1] and Lemma 2.4, we get

$$(5.36) \quad R^*(0; \rho) = \tau_X(\rho^{\mathbb{R}}; h)^{-1}.$$

Now assume that  $M \subset V$  is a lattice which is stable under  $\Gamma$  with respect to  $\rho$ . Let  $\mathcal{M}$  be the associated local system of free  $\mathbb{Z}$ -modules and let  $\mathcal{M}(\mathbb{R}) = \mathcal{M} \otimes \mathbb{R}$ . If  $\rho_\theta \not\cong \rho$ , we have  $H^*(X, \mathcal{M}(\mathbb{R})) = 0$ . Combining Theorem 2.3 and (5.34) we get

$$|R(0; \rho)| = \prod_{p=0}^3 |H^p(X, \mathcal{M})_{\text{tors}}|^{(-1)^p}.$$

Now assume that  $\rho_\theta = \rho$ . Then  $\text{rank } H^p(X, \mathcal{M}) = 2 \dim H^p(X, E_\rho)$ . Let  $\rho \neq 1$ . Then it follows from (5.35) that

$$\text{ord}_{s=0} R(s; \rho) = \sum_{p=1}^3 (-1)^p \text{rk } H^p(X, \mathcal{M}).$$

Using Theorem 2.3 and (5.36), we get

$$R^*(0; \rho) = R(\mathcal{M})^{-1} \prod_{p=0}^3 |H^p(X, \mathcal{M})_{\text{tors}}|^{(-1)^p}.$$

Finally, if  $\rho = 1$  it follows from [Mu2, Theorem 1.5] that the order of  $R(s; 1)$  at  $s = 0$  equals  $2 \dim H^1(X, \mathbb{R}) - 4$ .

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